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Are there irreducible spherically symmetric solutions in the SU(3) and SU(4) Yang–Mills theories?

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Abstract. In the SU(3) and SU(4) Yang-Mills theories in four-dimensional Euclidean space, there are no spherically symmetric instanton and meron solutions which belong to the full SU(3) or SU(4) algebra.

1. Introduction

The method of the phase factor of the standard differential loop (Gu 1981) is effective for studying spherically symmetric Yang-Mills theory. In terms of this method, we obtained all the possible asymptotic forms of SU(N) monopole solutions (Ma 1984) including reducible and irreducible ones. A solution is said to be reducible if it can be transformed by a gauge transformation into one which belongs to a Lie subalgebra. An irreducible solution belongs to the full Lie algebra. It is also in terms of this method that the general form of the SU(N) spherically symmetric gauge potential in four-dimensional Euclidean space is obtained (Ma and Xu 1984) as follows||

$$R_{\mu\nu}W_\nu(R^{-1}x)=\mathscr{D}(R^{-1})W_\mu(x)\mathscr{D}(R), \tag{1}$$

where R denotes a 4-rotation matrix and $\mathscr{D}(R)$ an N-dimensional representation of rotation group in four-dimensional Euclidean space (SO(4) groups). However, in the previous papers (Ma and Xu 1984), we use the generators of $\mathscr{D}(R)$ to form the gauge potential $W_\mu(x)$ so that only SO(4) embedding instanton and meron solutions were obtained. They are reducible ones. As we know, the irreducible cylindrically symmetric instanton solutions have been found in SU(3) Yang-Mills theory several years ago (Witten 1977, Bais and Weldon 1978). Do there exist any irreducible spherically symmetric instanton or meron solutions in SU(3) and SU(4) Yang-Mills theory? In this paper, we look for those irreducible solutions. Surprisingly enough, there are none of those solutions at all.

2. Spherically symmetric condition

In the central gauge (Gu 1981) a spherically symmetric gauge potential satisfies (1)

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||In this paper, the repeated μ, ν ... denotes summation of 1, 2, 3 and 4 and the repeated α, β ... summation of 1, 2, 3.

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and the consistent conditions

$$W_\mu(0)=0, \qquad x_\mu W_\mu(x)=0. \tag{2}$$

Through a similarity transformation, the representation $\mathscr{D}(R)$ of the SO(4) group can be expressed as a direct sum of the irreducible ones $\mathscr{D}^{jk}(R)$

$$\mathscr{D}(R)=\sum_{jk}^{\oplus}\mathscr{D}^{jk}(R), \tag{3}$$

where an additional index should be added if an irreducible representation appears more than once. In this paper, we do not restrict ourselves to form the gauge potential $W_\mu(x)$ by the generators of $\mathscr{D}(R)$, so the direct product form is not needed (see § 4). Note that the irreducibility of $\mathscr{D}(R)$ and that of the gauge potential $W_\mu(x)$ are different things because $W_\mu(x)$ need not have the same type of block matrix as $\mathscr{D}(R)$ has.

For any point x in the four-dimensional space, there exists a rotation R (not unique, see equations (18)-(20)) to turn x into the fourth axis:

$$x_0=Rx, \qquad x_0=(0,0,0,r), \tag{4}$$

$$r=(x_1^2+x_2^2+x_3^2+x_4^2)^{1/2}. \tag{5}$$

From (1), it follows that

$$W_\mu(x)=W_\mu(R^{-1}x_0)=\mathscr{D}(R^{-1})W_\nu(x_0)R_{\nu\mu}\mathscr{D}(R), \tag{6}$$

i.e. the spherically symmetric gauge potential $W_\mu(x)$ at any point x can be expressed by the potential $W_\mu(x_0)$ at the fourth axis. Furthermore, if one restricts the rotation R to be R_0 in (6) to keep the fourth axis invariant, $R_0^{-1}x_0=x_0$, i.e. R_0 is a rotation of the first three-dimensional space, (6) gives the relations among the different components of $W_\mu(x_0)$. We are going to discover these relations.

As we know, the generator $I_{\mu\nu}^{jk}$ of the irreducible representation $\mathscr{D}^{jk}(k)$ can be expressed as follows:

$$\begin{aligned} I_{\alpha\beta}^{jk}&=\varepsilon_{\alpha\beta\gamma}(L_\gamma^{jk}+K_\gamma^{jk}), \qquad & I_{\alpha 4}^{jk}&=L_\alpha^{jk}-K_\alpha^{jk},\\ L_\alpha^{jk}&=I_\alpha^j\times\mathbb{1}_{2k+1}, & K_\alpha^{jk}&=\mathbb{1}_{2j+1}\times I_\alpha^k, \end{aligned} \tag{7}$$

where I_α^j denotes the generator of the representation $\mathscr{D}^j(\mathrm{SU}(2))$

$$\begin{aligned} (I_3^j)_{ab}&=a\delta_{ab}, \qquad & I_\pm^j&\equiv I_1^j\pm \mathrm{i}I_2^j,\\ (I_+^j)_{ab}&=\delta_{a,b+1}\Gamma_a^j, & (I_-^j)_{ab}&=\delta_{a+1,b}\Gamma_b^j,\\ \Gamma_a^j&=[(j+a)(j-a+1)]^{1/2}, \end{aligned} \tag{8}$$

and $\mathbb{1}_n$ denotes a $n\times n$ unit matrix. $\mathscr{D}^{jk}$ is a $(2j+1)(2k+1)\times(2j+1)(2k+1)$ matrix and its row or column can be denoted by two indices a and b. Thus, the rows or columns of representation $\mathscr{D}(R)$ and gauge potential $W_\mu(x)$ can be denoted by four indices j, k, a, b:

$$\begin{aligned} a&=j,j-1,\ldots,-j,\\ b&=k,k-1,\ldots,-k. \end{aligned} \tag{9}$$

In terms of the explicit form of the generators of R_0 and $\mathscr{D}(R_0)$, where R_0 is the rotation in the first three-dimensional space, those relations are obtained as follows:

$$\begin{aligned} &W_1(x_0)_{jkab;\bar{j}\bar{k}\bar{a}\bar{b}}=[(\bar{a}+\bar{b})-(a+b)]^2W_1(x_0)_{jkab;\bar{j}\bar{k}\bar{a}\bar{b}},\\ &W_2(x_0)_{jkab;\bar{j}\bar{k}\bar{a}\bar{b}}=\mathrm{i}[(\bar{a}+\bar{b})-(a+b)]W_1(x_0)_{jkab;\bar{j}\bar{k}\bar{a}\bar{b}},\\ &[(\bar{a}+\bar{b})-(a+b)]W_3(x_0)_{jkab,\bar{j}\bar{k}\bar{a}\bar{b}}=0,\\ &[(\bar{a}+\bar{b})-(a+b)]W_4(x_0)_{jkab,\bar{j}\bar{k}\bar{a}\bar{b}}=0, \end{aligned} \tag{10a}$$

$$2\, W_1(x_0)_{jkab,\bar{j}\bar{k}(\bar{a}-1)\bar{b}} = \Gamma^{\bar{j}}_{\bar{a}} W_3(x_0)_{jkab,\bar{j}\bar{k}\bar{a}\bar{b}} + \Gamma^{\bar{k}}_{\bar{b}+1} W_3(x_0)_{jkab,\bar{j}\bar{k}(\bar{a}-1)(\bar{b}+1)} - \Gamma^{j}_{a} W_3(x_0)_{jk(a-1)b,\bar{j}\bar{k}(\bar{a}-1)\bar{b}} - \Gamma^{k}_{b} W_3(x_0)_{jka(b-1),\bar{j}\bar{k}(\bar{a}-1)\bar{b}}, \tag{10b}$$

$$2\, W_1(x_0)_{jk(a-1)b,\bar{j}\bar{k}\bar{a}\bar{b}} = \Gamma^{j}_{a} W_3(x_0)_{jkab;\bar{j}\bar{k}\bar{a}\bar{b}} + \Gamma^{k}_{b+1} W_3(x_0)_{jk(a-1)(b+1),\bar{j}\bar{k}\bar{a}\bar{b}} - \Gamma^{\bar{j}}_{\bar{a}} W_3(x_0)_{jk(a-1)b,\bar{j}\bar{k}(\bar{a}-1)\bar{b}} - \Gamma^{\bar{k}}_{\bar{b}} W_3(x_0)_{jk(a-1)b,\bar{j}\bar{k}\bar{a}(\bar{b}-1)},$$

$$\begin{aligned}
[(\Gamma^{j}_{a})^2+(\Gamma^{k}_{b})^2+(\Gamma^{\bar{j}}_{\bar{a}+1})^2+(\Gamma^{\bar{k}}_{\bar{b}+1})^2-2]\, W_3(x_0)_{jkab,\bar{j}\bar{k}\bar{a}\bar{b}}
&= \Gamma^{j}_{a+1}\Gamma^{\bar{j}}_{\bar{a}+1} W_3(x_0)_{jk(a+1)b,\bar{j}\bar{k}(\bar{a}+1)\bar{b}} + \Gamma^{j}_{a+1}\Gamma^{\bar{k}}_{\bar{b}+1} W_3(x_0)_{jk(a+1)b,\bar{j}\bar{k}\bar{a}(\bar{b}+1)} \\
&\quad + \Gamma^{k}_{b+1}\Gamma^{\bar{j}}_{\bar{a}+1} W_3(x_0)_{jka(b+1),\bar{j}\bar{k}(\bar{a}+1)\bar{b}} + \Gamma^{k}_{b+1}\Gamma^{\bar{k}}_{\bar{b}+1} W_3(x_0)_{jka(b+1),\bar{j}\bar{k}\bar{a}(\bar{b}+1)} \\
&\quad + \Gamma^{j}_{a}\Gamma^{\bar{j}}_{\bar{a}} W_3(x_0)_{jk(a-1)b,\bar{j}\bar{k}(\bar{a}-1)\bar{b}} + \Gamma^{j}_{a}\Gamma^{\bar{k}}_{\bar{b}} W_3(x_0)_{jk(a-1)b,\bar{j}\bar{k}\bar{a}(\bar{b}-1)} \\
&\quad + \Gamma^{k}_{b}\Gamma^{\bar{j}}_{\bar{a}} W_3(x_0)_{jka(b-1),\bar{j}\bar{k}(\bar{a}-1)\bar{b}} + \Gamma^{k}_{b}\Gamma^{\bar{k}}_{\bar{b}} W_3(x_0)_{jka(b-1),\bar{j}\bar{k}\bar{a}(\bar{b}-1)} \\
&\quad - \Gamma^{j}_{a}\Gamma^{k}_{b+1} W_3(x_0)_{jk(a-1)(b+1),\bar{j}\bar{k}\bar{a}\bar{b}} - \Gamma^{j}_{a+1}\Gamma^{k}_{b} W_3(x_0)_{jk(a+1)(b-1),\bar{j}\bar{k}\bar{a}\bar{b}} \\
&\quad - \Gamma^{\bar{j}}_{\bar{a}}\Gamma^{\bar{k}}_{\bar{b}+1} W_3(x_0)_{jkab,\bar{j}\bar{k}(\bar{a}-1)(\bar{b}+1)} - \Gamma^{\bar{j}}_{\bar{a}+1}\Gamma^{\bar{k}}_{\bar{b}} W_3(x_0)_{jkab,\bar{j}\bar{k}(\bar{a}+1)(\bar{b}-1)}.
\end{aligned} \tag{10c}$$

The consistency condition (2) requires

$$W_4(x_0)=0. \tag{10d}$$

It is obvious that the components with the subscripts such that $(j+k)-(\bar{j}+\bar{k})$ is equal to half of an odd integer must vanish.

The gauge field strength

$$G_{\mu\nu}(x)=\frac{\partial W_\nu(x)}{\partial x_\mu}-\frac{\partial W_\mu(x)}{\partial x_\nu}-ie(W_\mu W_\nu - W_\nu W_\mu) \tag{11}$$

satisfies the similar spherically symmetric condition when a rotation R is carried out:

$$R_{\mu\rho}R_{\nu\lambda}G_{\rho\lambda}(R^{-1}x)=\mathscr{D}(R^{-1})G_{\mu\nu}(x)\mathscr{D}(R), \tag{12a}$$

and from (4)

$$G_{\mu\nu}(x)=G_{\mu\nu}(R^{-1}x_0)=\mathscr{D}(R^{-1})G_{\rho\lambda}(x_0)R_{\rho\mu}R_{\lambda\nu}\mathscr{D}(R). \tag{12b}$$

Restricting R to be R_0 to keep the fourth axis invariant, we get the relations satisfied by the components of $G_{\mu\nu}(x_0)$. It is seen that the relations satisfied by $G_{23}(x_0)$, $G_{31}(x_0)$ and $G_{12}(x_0)$ are the same as those satisfied by $W_1(x_0)$, $W_2(x_0)$ and $W_3(x_0)$, as do $G_{14}(x_0)$, $G_{24}(x_0)$ and $G_{34}(x_0)$.

Let us discuss two simple examples. If $\mathscr{D}(R)=\mathscr{D}^{j0}(R)$, $k=\bar{k}=b=\bar{b}=0$, the subscripts $j=\bar{j}$ can be omitted and (10c) becomes

$$2(j^2+j-a^2-1)\, W_3(x_0)_{aa}=(j+a+1)(j-a)\, W_3(x_0)_{(a+1)(a+1)} + (j+a)(j-a+1)\, W_3(x_0)_{(a-1)(a-1)}.$$

The solution of this is

$$W_3(x_0)_{ab}=\delta_{ab}(a/j)\, W_3(x_0)_{jj}.$$

There is only one parameter $W_3(x_0)_{jj}/j\equiv c$. From (10), we obtain

$$\begin{aligned} W_\alpha(x_0)_{ab}&=cI^{j}_{\alpha}, \qquad \alpha=1,2,3,\\ W_4(x_0)_{ab}&=0. \end{aligned} \tag{13}$$

Therefore

$$W_\mu(x) = W_\mu(R^{-1}x_0) = c\mathscr{D}(R^{-1})I^j_\alpha R_{\alpha\mu}\mathscr{D}(R).$$

$W_\mu(x)$ belongs to the SU(2) subalgebra. In SU(3) gauge theory $\mathscr{D}(R)$ may be $3\mathscr{D}^{00}(R)$, $\mathscr{D}^{\frac{1}{2}0}\oplus\mathscr{D}^{00}(\mathscr{D}^{0\frac{1}{2}}\oplus\mathscr{D}^{00})$ and $\mathscr{D}^{10}(\mathscr{D}^{01})$. The first case is topologically trivial; in the second case, the solution must be the SU(2) embedding one because the components belonging to the different representations must be zero; and now we have shown that in the last case, the spherically symmetric solution must be the SU(2) embedding one. There are no irreducible spherically symmetric instanton and meron solutions in the SU(3) Yang-Mills theory.

The second example is that $\mathscr{D}(R) = \Sigma_j^\oplus \mathscr{D}^{j0}(R)$. We discuss the off-diagonal components with different j. From (10c), we get ($k = \bar{k} = 0$ have been omitted)

$$[j(j+1) + \bar{j}(\bar{j}+1) - 2a^2 - 2]\, W_3(x_0)_{ja,\bar{j}a}$$
$$= \Gamma^j_{a+1}\Gamma^{\bar{j}}_{a+1}\, W_3(x_0)_{j(a+1),\bar{j}(a+1)} + \Gamma^j_a\Gamma^{\bar{j}}_a\, W_3(x_0)_{j(a-1),\bar{j}(a-1)}.$$

Obviously, j and $\bar{j}$ must be integer or half-integer simultaneously. If $\bar{j} = \frac{1}{2}$ we obtain

$$[j(j+1) - \tfrac{7}{4}]\, W_3(x_0)_{j\frac{1}{2},\bar{j}\frac{1}{2}} = (j+\tfrac{1}{2})\, W_3(x_0)_{j-\frac{1}{2},\bar{j}-\frac{1}{2}},$$
$$[j(j+1) - \tfrac{7}{4}]\, W_3(x_0)_{j-\frac{1}{2},\bar{j}-\frac{1}{2}} = (j+\tfrac{1}{2})\, W_3(x_0)_{j\frac{1}{2},j\frac{1}{2}}.$$

The non-vanishing solution corresponds to

$$j = \tfrac{1}{2} \qquad \text{or} \qquad j = \tfrac{3}{2}. \tag{14}$$

In the case $\mathscr{D}^{\frac{1}{2}0}\oplus\mathscr{D}^{\frac{1}{2}0}$, $W_\alpha(x_0)_{ja,\bar{j}b} = C_{j\bar{j}}(\frac{1}{2}\sigma_\alpha)_{ab}$ which can be diagonalised into the block matrix by a global gauge transformation keeping $\mathscr{D}(R)$ invariant; therefore, it is an SU(2) embedding one. $\mathscr{D}^{\frac{3}{2}0}\oplus\mathscr{D}^{\frac{1}{2}0}$ will appear in SU(6) gauge theory. One can obtain the same result if some or all $\mathscr{D}^{j0}$ in $\mathscr{D}$ are replaced by $\mathscr{D}^{0k}$. In SU(4) gauge theory, there are two cases $\mathscr{D}(R) = \mathscr{D}^{\frac{1}{2}\frac{1}{2}}(R)$ and $\mathscr{D}(R) = \mathscr{D}^{10}(R)\oplus\mathscr{D}^{00}(R)$ where irreducible solutions may exist. $\mathscr{D}(R) = \mathscr{D}^{01}(R)\oplus\mathscr{D}^{00}(R)$ is a similar case to the latter. We will discuss them in the following two sections respectively.

3. $\mathscr{D}^{\frac{1}{2}\frac{1}{2}}$ case in SU(4) gauge theory

First, let us introduce the spherical coordinates in four-dimensional space,

$$x_1 = rs_1c_2s_3, \qquad x_2 = rs_1s_2s_3, \qquad x_3 = rc_1s_3, \qquad x_4 = rc_3, \tag{15}$$

where

$$\begin{aligned} s_1 &= \sin\theta, & s_2 &= \sin\varphi, & s_3 &= \sin\psi, \\ c_1 &= \cos\theta, & c_2 &= \cos\varphi, & c_3 &= \cos\psi. \end{aligned} \tag{16}$$

The unit vectors in the rectangular and spherical coordinates are related as follows:

$$\begin{aligned} &\hat{\theta} \equiv \hat{r}_1 = c_1c_2\hat{x}_1 + c_1s_2\hat{x}_2 - s_1\hat{x}_3, \\ &\hat{\varphi} \equiv \hat{r}_2 = -s_2\hat{x}_1 + c_2\hat{x}_2, \\ &\hat{\psi} \equiv \hat{r}_3 = s_1c_2c_3\hat{x}_1 + s_1s_2c_3\hat{x}_2 + c_1c_3\hat{x}_3 - s_3\hat{x}_4, \\ &\hat{r} \equiv \hat{r}_4 = s_1c_2s_3\hat{x}_1 + s_1s_2s_3\hat{x}_2 + c_1s_3\hat{x}_3 + c_3\hat{x}_4, \\ &\theta \equiv r_1, \qquad \varphi \equiv r_2, \qquad \psi \equiv r_3, \qquad r \equiv r_4. \end{aligned} \tag{17}$$

Define the rotation R as the following:

$$R = R_{34}(-\psi)R_{31}(\theta)R_{12}(\varphi)$$
$$= \begin{pmatrix} c_1c_2 & c_1s_2 & -s_1 & 0 \\ -s_2 & c_2 & 0 & 0 \\ s_1c_2c_3 & s_1s_2c_3 & c_1c_3 & -s_3 \\ s_1c_2s_3 & s_1s_2s_3 & c_1s_3 & c_3 \end{pmatrix}, \tag{18}$$

where $R_{\mu\nu}(\alpha)$ denotes the rotation with angle α along the $(\mu\nu)$ plane. R satisfies the following formulae

$$R\hat{r}_\mu = \hat{x}_\mu, \qquad \hat{x}_\mu R \equiv R^{-1}\hat{x}_\mu = \hat{r}_\mu,$$
$$R_{\mu\nu}\hat{x}_\nu = \hat{r}_\mu, \qquad \hat{r}_\mu R_{\mu\nu} = \hat{x}_\nu, \tag{19}$$

$$\boldsymbol{x} = x_\mu\hat{x}_\mu = r\hat{r}_4, \qquad \boldsymbol{x}_0 = r\hat{x}_4,$$
$$R\boldsymbol{x} = rR\hat{r}_4 = r\hat{x}_4 = \boldsymbol{x}_0, \tag{20}$$

i.e. this R satisfies (4). It is standard practice to obtain the formulae for the vectorial derivative

$$\hat{x}_\mu \frac{\partial S(x)}{\partial x_\mu} = \sum_{\mu=1}^{4} \hat{r}_\mu \frac{1}{H_\mu}\frac{\partial S(x)}{\partial r_\mu},$$
$$\frac{\partial A_\mu(x)}{\partial x_\mu} = \frac{1}{rs_1s_3}\frac{\partial s_1A_\theta(x)}{\partial\theta} + \frac{1}{rs_1s_3}\frac{\partial A_\varphi(x)}{\partial\varphi} + \frac{1}{rs_3^2}\frac{\partial s_3^2A_\psi(x)}{\partial\varphi} + \frac{1}{r^3}\frac{\partial r^3A_r(x)}{\partial r}, \tag{21}$$
$$\hat{x}_\mu\hat{x}_\nu\left(\frac{\partial A_\nu(x)}{\partial x_\mu} - \frac{\partial A_\mu(x)}{\partial x_\nu}\right) = \sum_{\mu\nu} \hat{r}_\mu\hat{r}_\nu \frac{1}{H_\mu H_\nu}\left(\frac{\partial H_\nu A_\nu(x)}{\partial r_\mu} - \frac{\partial H_\mu A_\mu(x)}{\partial r_\nu}\right),$$

where

$$H_1 = rs_3, \qquad H_2 = rs_1s_3, \qquad H_3 = r, \qquad H_4 = 1. \tag{22}$$

The generators of $\mathscr{D}^{\frac{1}{2}\frac{1}{2}}(R)$ are

$$I_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}(L_\gamma + K_\gamma), \qquad I_{\alpha 4} = L_\alpha - K_\alpha,$$
$$L_\alpha = \tfrac{1}{2}\sigma_\alpha \times \mathbb{1}_2, \qquad K_\alpha = \mathbb{1}_2 \times \tfrac{1}{2}\sigma_\alpha. \tag{23}$$

Now we introduce the bases of matrices

$$\Sigma_1 = \tfrac{1}{2}(\sigma_2 \times \sigma_3 - \sigma_3 \times \sigma_2),$$
$$\Sigma_2 = \tfrac{1}{2}(\sigma_3 \times \sigma_1 - \sigma_1 \times \sigma_3),$$
$$\Sigma_3 = \tfrac{1}{2}(\sigma_1 \times \sigma_2 - \sigma_2 \times \sigma_1), \tag{24}$$
$$\Sigma_0 = \sum_{\alpha=1}^{3} \sigma_\alpha \times \sigma_\alpha.$$

Since $j = k = \bar{j} = \bar{k} = \frac{1}{2}$ for $\mathscr{D}^{\frac{1}{2}\frac{1}{2}}(R)$, we may omit the subscripts j, k, $\bar{j}$ and $\bar{k}$ and use $\pm$ to denote the values of a and b. Now, the independent equalities from (10c) are as follows:

$$W_3(x_0)_{-+-+} + W_3(x_0)_{+-+-} = 0,$$
$$W_3(x_0)_{-++-} + W_3(x_0)_{+--+} = 0,$$
$$W_3(x_0)_{++++} + W_3(x_0)_{----} = 0.$$

That means

$$W_3(x_0)=AI_{34}+BI_{12}+C\Sigma_3, \tag{25a}$$

and from (10)

$$\begin{aligned} W_1(x_0)&=AI_{14}+BI_{23}+C\Sigma_1,\\ W_2(x_0)&=AI_{24}+BI_{31}+C\Sigma_2. \end{aligned} \tag{25b}$$

Similarly,

$$W_4(x_0)=D\Sigma_0, \tag{25c}$$

where A, B, C and D are the functions of $x_0=r$. Obviously, $D=0$ because of the consistent condition (2). In order to separate the angular part from the radial one, we introduce the following bases for the vectors

$$\begin{aligned} V^1(x_0)&=\hat{x}_\alpha I_{\alpha 4}, \qquad V^2(x_0)=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}\hat{x}_\alpha I_{\beta\gamma},\\ V^3(x_0)&=\hat{x}_\alpha\Sigma_\alpha, \qquad V^4(x_0)=\hat{x}_4\Sigma_0, \end{aligned} \tag{26a}$$

and

$$\begin{aligned} V^1(x)&=\hat{r}_\alpha\mathscr{D}(R)^{-1}I_{\alpha 4}\mathscr{D}(R),\\ V^2(x)&=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha\mathscr{D}(R)^{-1}I_{\beta\gamma}\mathscr{D}(R),\\ V^3(x)&=\hat{r}_\alpha\mathscr{D}(R)^{-1}\Sigma_\alpha\mathscr{D}(R),\\ V^4(x)&=\hat{r}_4\mathscr{D}(R)^{-1}\Sigma_0\mathscr{D}(R), \end{aligned} \tag{26b}$$

where R is defined in (18). The explicit forms of $V^\mu(x)$ can be calculated easily. Now, the gauge potential $W_\mu(x)$ can be expressed as

$$W_\mu(x)\hat{x}_\mu=r\phi_1(r)V^1(x)+r\phi_2(r)V^2(x)+r\phi_3(r)V^3(x). \tag{27}$$

In order to express the gauge field strength $G_{\mu\nu}(x)$, we introduce the bases for antisymmetric tensors

$$\begin{aligned} \mathbf{T}^{10}(x)&=\varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha\hat{r}_\beta\mathscr{D}(R)^{-1}I_{\gamma 4}\mathscr{D}(R),\\ \mathbf{T}^{20}(x)&=\hat{r}_\alpha\hat{r}_\beta\mathscr{D}(R)^{-1}I_{\alpha\beta}\mathscr{D}(R),\\ \mathbf{T}^{30}(x)&=\varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha\hat{r}_\beta\mathscr{D}(R)^{-1}\Sigma_\gamma\mathscr{D}(R),\\ \mathbf{T}^{01}(x)&=(\hat{r}_\alpha\hat{r}_4-\hat{r}_4\hat{r}_\alpha)\mathscr{D}(R)^{-1}I_{\alpha 4}\mathscr{D}(R),\\ \mathbf{T}^{02}(x)&=\tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}(\hat{r}_\alpha\hat{r}_4-\hat{r}_4\hat{r}_\alpha)\mathscr{D}(R)^{-1}I_{\beta\gamma}\mathscr{D}(R),\\ \mathbf{T}^{03}(x)&=(\hat{r}_\alpha\hat{r}_4-\hat{r}_4\hat{r}_\alpha)\mathscr{D}(R)^{-1}\Sigma_\alpha\mathscr{D}(R). \end{aligned} \tag{28}$$

When one calculates $G_{\mu\nu}(x)$ and substitutes them into the Yang-Mills equation

$$\partial G_{\mu\nu}(x)/\partial x_\mu-ie(W_\mu(x)G_{\mu\nu}(x)-G_{\mu\nu}(x)W_\mu(x))=0 \tag{29}$$

it is easier to calculate the commutators at the fourth axis and the derivative near the fourth axis (after derivative, let ψ tend to zero). Through the straightforward calculation, we obtain

$$\begin{aligned} G_{\mu\nu}(x)\hat{x}_\mu\hat{x}_\nu&=2\phi_2(-1+e\phi_1r^2)\mathbf{T}^{10}+[-2\phi_1+er^2(\phi_1^2+\phi_2^2+\phi_3^2)]\mathbf{T}^{20}\\ &\quad+2er^2\phi_2\phi_3\mathbf{T}^{30}-\sum_{\alpha=1}^{3}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r^2\phi_\alpha\right)\mathbf{T}^{0\alpha}, \end{aligned} \tag{30}$$

and

$$2\tau\ddot{\phi}_1+6\dot{\phi}_1+e(3\phi_1^2+3\phi_2^2+\phi_3^2)-e^2\tau\phi_1(\phi_1^2+3\phi_2^2+\phi_3^2)=0, \tag{31a}$$

$$2\tau\ddot{\phi}_2+6\dot{\phi}_2+6e\phi_1\phi_2-e^2\tau\phi_2(3\phi_1^2+\phi_2^2+3\phi_3^2)=0, \tag{31b}$$

$$2\tau\ddot{\phi}_3+6\dot{\phi}_3+(2/\tau)\phi_3+2e\phi_1\phi_3-e^2\tau\phi_3(\phi_1^2+3\phi_2^2+\phi_3^2)=0, \tag{31c}$$

$$e\tau\phi_3\dot{\phi}_1-e\tau\phi_1\dot{\phi}_3+\dot{\phi}_3+(1/\tau)\phi_3=0, \tag{31d}$$

where a dot denotes the derivative with respect to $\tau=r^2$.

If $\phi_3=0$, $W_\mu(x)$ reduces to the SO(4) embedding one discussed previously (Ma and Xu 1984). Now, we are interested in the solutions with $\phi_3\neq 0$. From (31d), we obtain

$$\phi_3=c_1(\phi_1-1/e\tau). \tag{32}$$

Substituting into (31c), (31c) becomes the same as (31a). Now, we have to solve the three-coupled differential equations (31a), (31b) and (32). If ϕ_1 and ϕ_2 were finite near the origin, $\phi_3\sim 1/e\tau$ and (31a) and (31b) could not be satisfied. Therefore, there are no spherically symmetric irreducible instanton solutions in the SU(4) gauge theory. According to the forms of the equations, we are going to look for the meron solutions in the following form:

$$\phi_1=A/e\tau, \qquad \phi_2=B/e\tau, \qquad \phi_3=C/e\tau, \tag{33}$$

where A, B and C are constants, and

$$C=0 \qquad \text{if} \qquad A=1. \tag{34}$$

Now, equations (31a) and (31b) become

$$(A-1)[A(2-A)-3B^2-C^2]=0,$$

and

$$B(-2+6A-3A^2-B^2-3C^2)=0.$$

If $B\neq 0$, we have

$$B^2=\tfrac{1}{4}, \qquad C^2=\tfrac{1}{4}-(A-1)^2. \tag{35}$$

If $B=0$, we have

$$C^2=1-(A-1)^2. \tag{36}$$

Therefore, we obtain the following SU(4) meron solutions

$$W_\mu(x)\hat{x}_\mu=\frac{V^1(x)}{er}+\frac{1}{2er}(\cos\omega V^1(x)+\sin\omega V^3(x))\pm\frac{1}{2er}V^2(x) \tag{37a}$$

$$W_\mu(x)\hat{x}_\mu=\frac{1}{er}(V^1(x)+\cos\omega V^1(x)+\sin\omega V^3(x)) \qquad -\pi<\omega\leqslant\pi. \tag{37b}$$

Since the meron solutions are singular at the origin, we have to check whether they are gauge equivalent to the reducible ones through some gauge transformations which may be singular at the origin.

It is easy to check that

$$U_0(\omega)\equiv\exp(\tfrac{1}{4}i\omega\Sigma_0)=\begin{pmatrix}\exp(\frac{1}{4}i\omega) & & \\ & \exp(-\frac{1}{4}i\omega)(\cos\frac{1}{2}\omega+i\sigma_1\sin\frac{1}{2}\omega) & \\ & & \exp(\frac{1}{4}i\omega)\end{pmatrix} \tag{38}$$

and

$$\begin{aligned} &U_0(\omega) I_{\alpha\beta} U_0(\omega)^{-1} = I_{\alpha\beta}, \\ &U_0(\omega) I_{\alpha 4} U_0(\omega)^{-1} = \cos \omega I_{\alpha 4} - \sin \omega \Sigma_\alpha, \qquad \alpha = 1, 2, 3. \\ &U_0(\omega) \Sigma_\alpha U_0(\omega)^{-1} = \sin \omega I_{\alpha 4} + \cos \omega \Sigma_\alpha, \end{aligned} \tag{39}$$

Define a gauge transformation which is singular at the origin:

$$U(\omega) = \mathscr{D}^{-1}(R) U_0(\omega) \mathscr{D}(R), \tag{40}$$

where R is defined in (18) and $\mathscr{D} = \mathscr{D}^{\frac{1}{2}\frac{1}{2}}$. It follows that

$$\begin{aligned} &U(\omega) V^2(x) U(\omega)^{-1} = V^2(x), \\ &U(\omega) V^1(x) U(\omega)^{-1} = \cos \omega V^1(x) - \sin \omega V^3(x), \\ &U(\omega) V^3(x) U(\omega)^{-1} = \sin \omega V^1(\alpha) + \cos \omega V^3(x), \end{aligned}$$

and

$$U(\omega)[\cos \omega V^1(x) + \sin \omega V^3(x)]\, U(\omega)^{-1} = V^1(x). \tag{41}$$

In terms of the formulae

$$\begin{aligned} &\mathscr{D}(R) = \mathscr{D}(I_{34}, -\psi) \mathscr{D}(I_{31}, \theta) \mathscr{D}(I_{12}, \varphi), \\ &[(\partial/\partial\theta) \mathscr{D}(I_{\mu\nu}, \theta)] \mathscr{D}(I_{\mu\nu}, -\theta) = \mathrm{i} I_{\mu\nu}, \qquad \text{(no summation of } \mu, \nu) \\ &\mathscr{D}^{\frac{1}{2}\frac{1}{2}}(R) = \mathscr{D}^{\frac{1}{2}0}(R) \times \mathscr{D}^{0\frac{1}{2}}(R), \end{aligned} \tag{42}$$

we obtain

$$-\frac{\mathrm{i}}{e}[\partial_\mu U(\omega)] U(\omega)^{-1} \hat{x}_\mu = \frac{1}{er}[(1 - \cos \omega) V^1(x) + \sin \omega V^3(x)]. \tag{43}$$

Therefore, under the gauge transformation, the solutions (37*a*,*b*) become

$$W_\mu(x) \hat{x}_\mu \to \frac{3}{2er} V^1(x) \pm \frac{1}{2er} V^2(x), \tag{44a}$$

$$W_\mu(x) \hat{x}_\mu \to \frac{2}{er} V^1(x). \tag{44b}$$

Obviously, they are the SO(4) embedding solutions (Ma and Xu 1984). In fact, the term $r\phi_3(r) V^3(x)$ in that solution (33) can be removed by the singular gauge transformation $U(\omega)$:

$$\begin{aligned} W_\mu(x) \hat{x}_\mu \to \frac{1}{er} \{ &[A \cos \omega + C \sin \omega + (1 - \cos \omega)] V^1(x) \\ &+ B V^2(x) + (-A \sin \omega + C \cos \omega + \sin \omega) V^3(x) \} \end{aligned}$$

where ω is chosen to satisfy

$$-A \sin \omega + C \cos \omega + \sin \omega = 0.$$

Therefore, those types of meron solutions are the SO(4) embedding ones if the singular gauge transformation $U(\omega)$ is allowable.

4. $\mathscr{D}^{10} \oplus \mathscr{D}^{00}$ case in SU(4) gauge theory

Now, we turn to the case of $\mathscr{D}(R) = \mathscr{D}^{10}(R) \oplus \mathscr{D}^{00}(R)$. The row and column can be denoted by 1, 0, −1 and $\bar{0}$; the last $\bar{0}$ belongs to the representation $\mathscr{D}^{00}(R)$. From

(10c), we get

$$W_3(x_0)_{00}=0, \qquad W_3(x_0)_{11}=-W_3(x_0)_{-1-1}, \qquad W_3(x_0)_{\bar{0}\bar{0}}=0,$$

and there is no restriction for $W_3(x_0)_{0\bar{0}}$. Therefore

$$W_3(x_0)=\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & B+\mathrm{i}C \\ 0 & 0 & -A & 0 \\ 0 & B-\mathrm{i}C & 0 & 0 \end{pmatrix},$$

$$W_1(x_0)=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & A & 0 & -B-\mathrm{i}C \\ A & 0 & A & 0 \\ 0 & A & 0 & B+\mathrm{i}C \\ -B+\mathrm{i}C & 0 & B-\mathrm{i}C & 0 \end{pmatrix}, \tag{45}$$

$$W_2(x_0)=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i}A & 0 & \mathrm{i}B-C \\ \mathrm{i}A & 0 & -\mathrm{i}A & 0 \\ 0 & \mathrm{i}A & 0 & \mathrm{i}B-C \\ -\mathrm{i}B-C & 0 & -\mathrm{i}B-C & 0 \end{pmatrix}, \qquad W_4(x_0)=0.$$

We introduce the bases of matrices

$$J_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J_3=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad L_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad L_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{46}$$

$$K_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad K_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \qquad K_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},$$

$$J_4=\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

They satisfy the following commutative relations

$$\begin{aligned} &[J_\alpha, J_\beta] = [L_\alpha, L_\beta] = [K_\alpha, K_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma} J_\gamma, \\ &[L_\alpha, J_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma} L_\gamma, \qquad [K_\alpha, J_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma} K_\gamma. \end{aligned} \tag{47}$$

The generators of $\mathscr{D}^{10}(R)$ are

$$I_{12} = I_{34} = J_3, \qquad I_{23} = I_{14} = J_1, \qquad I_{31} = I_{24} = J_2. \tag{48}$$

According to (45), the bases for vectors are

$$\begin{aligned} &\boldsymbol{V}^1(x_0) = \hat{x}_\alpha J_\alpha, \qquad \boldsymbol{V}^2(x_0) = \hat{x}_\alpha L_\alpha, \\ &\boldsymbol{V}^3(x_0) = \hat{x}_\alpha K_\alpha, \qquad \boldsymbol{V}^4(x_0) = \hat{x}_4 J_4, \end{aligned} \tag{49a}$$

and

$$\begin{aligned} &\boldsymbol{V}^1(x) = \hat{r}_\alpha \mathscr{D}(R^{-1}) J_\alpha \mathscr{D}(R) = \hat{r}_\alpha \bar{R}_{\alpha\beta} J_\beta, \\ &\boldsymbol{V}^2(x) = \hat{r}_\alpha \mathscr{D}(R^{-1}) L_\alpha \mathscr{D}(R) = \hat{r}_\alpha \bar{R}_{\alpha\beta} L_\beta, \\ &\boldsymbol{V}^3(x) = \hat{r}_\alpha \mathscr{D}(R^{-1}) K_\alpha \mathscr{D}(R) = \hat{r}_\alpha \bar{R}_{\alpha\beta} K_\beta, \\ &\boldsymbol{V}^4(x) = \hat{r}_4 \mathscr{D}(R^{-1}) J_4 \mathscr{D}(R) = \hat{r} J_4, \end{aligned} \tag{49b}$$

where R is defined in (18) and

$$\bar{R} = \begin{pmatrix} c_1 c_2 c_3 + s_2 s_3 & c_1 s_2 c_3 - c_2 s_3 & -s_1 c_3 \\ c_1 c_2 s_3 - s_2 c_3 & c_1 s_2 s_3 + c_2 c_3 & -s_1 s_3 \\ s_1 c_2 & s_1 s_2 & c_1 \end{pmatrix}. \tag{50}$$

Thus, the general form of the gauge potential with $\mathscr{D} = \mathscr{D}^{10} \oplus \mathscr{D}^{00}$ can be expressed as

$$W_\mu(x)\hat{x}_\mu = r\phi_1(r)\boldsymbol{V}^1(x) + r\phi_2(r)\boldsymbol{V}^2(x) + r\phi_3(r)\boldsymbol{V}^3(x). \tag{51}$$

The gauge field strength is

$$\begin{aligned} G_{\mu\nu}(x)\hat{x}_\mu \hat{x}_\nu = {} & [-2\phi_1 + e(\phi_1^2 + \phi_2^2 + \phi_3^2)]\mathbf{T}^{10} + (-2\phi_2 + 2er^2\phi_1\phi_2)\mathbf{T}^{20} \\ & + (-2\phi_3 + 2er^2\phi_1\phi_3)\mathbf{T}^{30} - \sum_{\alpha=1}^{3}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r^2\phi_\alpha\right)\mathbf{T}^{0\alpha} \end{aligned} \tag{52}$$

where the bases $\mathbf{T}$ for the antisymmetric tensors are as follows:

$$\begin{aligned} &\mathbf{T}^{10}(x) = \varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha \hat{r}_\beta \mathscr{D}(R^{-1}) J_\gamma \mathscr{D}(R) = \varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha \hat{r}_\beta \bar{R}_{\gamma\delta} J_\delta, \\ &\mathbf{T}^{01}(x) = (\hat{r}_\alpha \hat{r}_4 - \hat{r}_4 \hat{r}_\alpha)\mathscr{D}(R^{-1}) J_\alpha \mathscr{D}(R) = (\hat{r}_\alpha \hat{r}_4 - \hat{r}_4 \hat{r}_\alpha)\bar{R}_{\alpha\beta} J_\beta, \\ &\mathbf{T}^{20}(x) = \varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha \hat{r}_\beta \bar{R}_{\gamma\delta} L_\delta, \qquad \mathbf{T}^{30}(x) = \varepsilon_{\alpha\beta\gamma}\hat{r}_\alpha \hat{r}_\beta \bar{R}_{\gamma\delta} K_\delta, \\ &\mathbf{T}^{02}(x) = (\hat{r}_\alpha \hat{r}_4 - \hat{r}_4 \hat{r}_\alpha)\bar{R}_{\alpha\beta} L_\beta, \quad \mathbf{T}^{03}(x) = (\hat{r}_\alpha \hat{r}_4 - \hat{r}_4 \hat{r}_\alpha)\bar{R}_{\alpha\beta} K_\beta. \end{aligned} \tag{53}$$

Substituting them into the Yang-Mills equation (29), we get

$$\begin{aligned} &2\tau\ddot{\phi}_1 + 6\dot{\phi}_1 + 3e(\phi_1^2 + \phi_2^2 + \phi_3^2) - e^2\tau\phi_1(\phi_1^2 + 3\phi_2^2 + 3\phi_3^2) = 0, \\ &2\tau\ddot{\phi}_2 + 6\dot{\phi}_2 + 6e\phi_1\phi_2 - e^2\tau\phi_2(3\phi_1^2 + \phi_2^2 + \phi_3^2) = 0, \\ &2\tau\ddot{\phi}_3 + 6\dot{\phi}_3 + 6e\phi_1\phi_3 - e^2\tau\phi_3(3\phi_1^2 + \phi_2^2 + \phi_3^2) = 0, \\ &\phi_2\dot{\phi}_3 = \phi_3\dot{\phi}_2, \end{aligned} \tag{54}$$

where a dot denotes the derivatives with respect to $\tau = r^2$. From the fourth equation, we get

$$\phi_3 = C\phi_2, \tag{55}$$

and then the second equation is not independent of the third one. Generally, taking $c = \tan\omega$, we introduce

$$\phi_4(r) = \frac{1}{\cos\omega}\phi_2(r) = \frac{1}{\sin\omega}\phi_3(r), \tag{56}$$

$$\phi_4^2(r) = \phi_2^2(r) + \phi_3^2(r). \tag{57}$$

They satisfy the following equation

$$\begin{aligned} &2\tau\ddot{\phi}_1 + 6\dot{\phi}_1 + 3e(\phi_1^2 + \phi_4^2) - e^2\tau\phi_1(\phi_1^2 + 3\phi_4^2) = 0, \\ &2\tau\ddot{\phi}_4 + 6\dot{\phi}_4 + 6e\phi_1\phi_4 - e^2\tau\phi_4(3\phi_1^2 + \phi_4^2) = 0. \end{aligned} \tag{58}$$

This is the equation which we discussed previously where we obtained the instanton and meron solutions

$$\begin{aligned} &\phi_1 + \phi_4 = \frac{2}{e(\tau + a^2)}, \qquad \phi_1 - \phi_4 = \frac{2}{e(\tau + b^2)}, \\ &a^2 > 0, \qquad b^2 > 0, \end{aligned} \tag{59a}$$

and

$$\phi_1 \pm \phi_4 = M_\pm / e\tau, \qquad M_\pm = 0, 1, 2. \tag{59b}$$

However, all these solutions are SO(4) embedding solutions because

$$\hat{x}_\mu W_\mu(x) = r\phi_1(r)\boldsymbol{V}^1(x) + r\phi_4(r)(\cos\omega\boldsymbol{V}^2(x) + \sin\omega\boldsymbol{V}^3(x)), \tag{60}$$

where $\boldsymbol{V}^1(x)$ and $(\cos\omega\boldsymbol{V}^2(x) + \sin\omega\boldsymbol{V}^3(x))$ are the linear combinations of the six matrices $\boldsymbol{J}$ and $\boldsymbol{L}' \equiv (\cos\omega\boldsymbol{L} + \sin\omega\boldsymbol{K})$ which belong to the SO(4) subalgebra. In fact,

$$\begin{aligned} &X^{-1}[\tfrac{1}{2}(\boldsymbol{J} + \boldsymbol{L}')]X = \tfrac{1}{2}\boldsymbol{\sigma} \times \mathbb{1}_2, \\ &X^{-1}[\tfrac{1}{2}(\boldsymbol{J} - \boldsymbol{L}')]X = \mathbb{1}_2 \times \tfrac{1}{2}\boldsymbol{\sigma}, \end{aligned} \tag{61}$$

where

$$X = \begin{pmatrix} e^{i\omega/2} & 0 & 0 & 0 \\ 0 & (1/\sqrt{2})\,e^{i\omega/2} & (1/\sqrt{2})\,e^{i\omega/2} & 0 \\ 0 & 0 & 0 & e^{i\omega/2} \\ 0 & (1/\sqrt{2})\,e^{-i\omega/2} & (-1/\sqrt{2})\,e^{-i\omega/2} & 0 \end{pmatrix}.$$

Under this global gauge transformation X,

$$X^{-1}(\mathscr{D}^{10} \oplus \mathscr{D}^{00})X = \mathscr{D}^{\frac{1}{2}0} \times \mathscr{D}^{\frac{1}{2}0}, \tag{62}$$

$$\begin{aligned} &X^{-1}W_\mu(x)X = r(\phi_1(r) + \phi_4(r))\boldsymbol{V}_+(x) + r(\phi_1(r) - \phi_4(r))\boldsymbol{V}_-(x), \\ &\boldsymbol{V}_+(x) = \hat{r}_\alpha[\mathscr{D}^{\frac{1}{2}0}(R^{-1})\tfrac{1}{2}\sigma_\alpha\mathscr{D}^{\frac{1}{2}0}(R)] \times \mathbb{1}_2 = \hat{r}_\alpha\bar{R}_{\alpha\beta}(\tfrac{1}{2}\sigma_\beta \times \mathbb{1}_2), \\ &\boldsymbol{V}_-(x) = \mathbb{1}_2 \times \hat{r}_\alpha[\mathscr{D}^{\frac{1}{2}0}(R^{-1})\tfrac{1}{2}\sigma_\alpha\mathscr{D}^{\frac{1}{2}0}(R)] = \hat{r}_\alpha\bar{R}_{\alpha\beta}(\mathbb{1}_2 \times \tfrac{1}{2}\sigma_\beta), \end{aligned} \tag{63}$$

i.e. $W_\mu(x)$ is equivalent to the combination of two SU(2) instanton or meron solutions.

In summary, we have looked for the irreducible spherically symmetric instanton and meron solutions. It is very surprising to us that there are no irreducible spherically symmetric instanton and meron solutions in SU(3) and SU(4) Yang-Mills theories at all, even though the irreducible cylindrically symmetric instanton solutions have been found in SU(3) Yang-Mills theory.

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